LECTURE 4 CONSUMPTION

1-52

CONSUMPTION: MOTIVATION

Consumption is the sole end and purpose of all production.

Adam Smith

Consumption: Macro vs. Micro

- Micro:
  - Apples versus Oranges (vs. TVs vs. trips to Hawaii, etc)
  - Normal goods, luxury goods, inferior goods, Giffen goods, etc.
  - Slutsky equation, Shephard's lemma, Roy's identity, etc.
  - Demand systems: CES, AIDS, Translog, etc.
- Macro:
  - Behavior of overall individual consumption
  - Response of consumption to changes in income and interest rates
  - Consumption and saving over the life-cycle
  - Response of economy-wide aggregate consumption to income and interest rates

A two-period model of consumption

Partial equilibrium case

- Partial equilibrium: study problem of individual in isolation taking as given prices. $\bar{p}$
- Two time periods $t = 1$ and $t = 2$.
- Consumption c_1 and c_2, income y_1 and y_2.
- Utility function

$$u(c_1) + \beta u(c_2)$$

with u strictly increasing, concave, discount factor $0 < \beta < 1$

Households solves

$$h = u(c_1) + \beta u(c_2) - \lambda_1 (y_1 - c_1 - a) + \lambda_2 (y_2 - c_2 + (1+r)a)$$

$\{c_1\}$: $u'(c_1) = \lambda_1$ ① $\{c_2\}$: $\beta u'(c_2) = \lambda_2$ ②

$\{a\}$: $\lambda_1 = \lambda_2 (1+r)$ ③

①②③ $\Rightarrow$ $u'(c_1) = \beta(1+r) u'(c_2)$

$$\max_{c_1, c_2, a} u(c_1) + \beta u(c_2)$$

$$s.t. \quad c_1 + a = y_1$$

$$c_2 = y_2 + (1+r)a$$

- c1, c2: consumption at t = 1 and t = 2.
- y1, y2: income at t = 1 and t = 2.
- r : interest rate (for now exogenously given)
- a: saving (a can be negative)

Partial equilibrium case

- Implicit assumption: can borrow and save as much you want at rate r.
- Combine the budget constraints:

$$a = \frac{c_2}{1+r} - \frac{y_2}{1+r}$$

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

现值

- Households problem is then

β决定下一期效用对现在的重要性

$$\max_{c_1, c_2} u(c_1) + \beta u(c_2)$$

$$s.t. \quad c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

$$F(c_1,c_2) = u(c_1) + \beta u(c_2) + \lambda\left(c_1 + \frac{c_2}{1+r} - y_1 - \frac{y_2}{1+r}\right)$$

$$\frac{\partial F}{\partial c_1} = u'(c_1) + \lambda = 0$$

- Optimal condition $\frac{\partial F}{\partial c_2} = \beta u'(c_2) + \frac{\lambda}{1+r} = 0$

$$\beta u'(c_2) = \frac{u'(c_1)}{1+r}$$

$$\Rightarrow u'(c_1) = \beta(1+r)u'(c_2).$$

$$u'(c_1) = \beta(1+r)u'(c_2)$$

- A key equation named "Euler Equation" after Leonard Euler. An intertemporal optimality condition.
- Related question: when to borrow/save? what if r increases?

Setup: log utility, two periods, bonds, bond price q_t, exogenous income Y_t, Y_{t+1} (more on the bond)

$$\max_{C_t, B_t, C_{t+1}} lnC_t + \beta lnC_{t+1}$$

$$s.t. \quad C_t + q_t B_t \leq Y_t$$

$$C_{t+1} \leq Y_{t+1} + B_t$$

Form a Lagrangian:

$$\mathcal{L} = lnC_t + \beta lnC_{t+1}$$

$$+ \lambda_1 (Y_t - C_t - q_t B_t)$$

$$+ \lambda_2 (Y_{t+1} + B_t - C_{t+1})$$

EQUILIBRIUM

- $B_t = 0$: only one type of agents
- $C_t = Y_t, C_{t+1} = Y_{t+1}$
- $$q_t = \beta \frac{Y_t}{Y_{t+1}}$$

$q_t = \beta(1+r) q_{t+1}$

$\beta \frac{Y_t}{Y_{t+1}} = \beta(1+r) q_{t+1}$

$\frac{Y_t}{1+r} = Y_{t+1} \cdot q_{t+1} \Rightarrow (q_{t+1} = 1)$

- Permanent income hypothesis does not work in this setting. The consumption smoothing channel is shut down for there is no essential saving tools in the model.
- Rewrite $1 + r_t = \frac{1}{q_t}$

$$1 + r_t = \frac{Y_{t+1}}{\beta Y_t}$$

- Fix bond supply at 0, an increase in Y_{t+1} reduces bond demand, drives down the bond price (increases the real rate r_t)

General equilibrium case: A stochastic case

- Two states: a high state $Y_{t+1,1}$ with probability p, a low state $Y_{t+1,2}$ with probability $1-p$.

$$\max_{C_t, B_t, C_{t+1,1}, C_{t+1,2}} lnC_t + \beta p lnC_{t+1,1} + \beta(1-p) lnC_{t+1,2}$$

$$\begin{aligned} s.t. \quad & C_t + q_t B_t \leq Y_t \\ & C_{t+1,1} \leq Y_{t+1,1} + B_t \\ & C_{t+1,2} \leq Y_{t+1,2} + B_t \end{aligned}$$

Form a Lagrangian:

$$\begin{aligned} \mathcal{L} = & lnC_t + \beta p lnC_{t+1,1} + \beta(1-p) lnC_{t+1,2} \\ & + \lambda_1 (Y_t - C_t - q_t B_t) \\ & + \lambda_2 (Y_{t+1,1} + B_t - C_{t+1,1}) \\ & + \lambda_3 (Y_{t+1,2} + B_t - C_{t+1,2}) \end{aligned}$$

FOCs

$$\{C_t\}: \qquad \frac{1}{C_t} = \lambda_1$$

$$\{C_{t+1,1}\}: \qquad p\beta\frac{1}{C_{t+1,1}} = \lambda_2$$

$$\{C_{t+1,2}\}: \qquad (1-p)\beta\frac{1}{C_{t+1,2}} = \lambda_3$$

$$\{B_t\}: \qquad q_t\lambda_1 = \lambda_2 + \lambda_3$$

We get

$$q_t\frac{1}{C_t} = \beta(p\frac{1}{C_{t+1,1}} + (1-p)\frac{1}{C_{t+1,2}})$$

of which the right-hand side is the expected marginal utility of future consumption.

$$q_t\frac{1}{C_t} = \beta E_t\frac{1}{C_{t+1}}$$

Implications of uncertainty

A mean-preserving increase in uncertainty will increase agents demand for smoothing (bond, saving). It will drives up the bond price, thus decrease r_t.

General equilibrium case: Two types of agents

- They have identical preferences, but differ somehow in their endowments.
- Log utility, same size of two types of agents.
- Endowment

type 1 $$(Y_{1,t}, Y_{1,t+1}) = (1, 0)$$

type 2 $$(Y_{2,t}, Y_{2,t+1}) = (0, 1)$$

- $$\frac{1}{C_{i,t}} = \beta \frac{1}{q_t} \frac{1}{C_{i,t+1}} \qquad i = 1, 2$$

- $$\frac{C_{1,t+1}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{2,t}}$$
$$\frac{C_{2,t}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{1,t+1}}$$

Conditions

$$B_{1,t} + B_{2,t} = 0$$

$$C_{1,t} + C_{2,t} = Y_{1,t} + Y_{2,t} = 1$$

$$C_{1,t+1} + C_{2,t+1} = Y_{1,t+1} + Y_{2,t+1} = 1$$

$$\frac{1 - C_{2,t}}{C_{2,t}} = \frac{1 - C_{2,t+1}}{C_{2,t+1}}$$

$C_{1,t} > C_{2,t}$

$$\begin{cases} C_{2,t} = C_{2,t+1} \\ C_{1,t} = C_{1,t+1} \end{cases}$$

两期消费相等

Plug back into the Euler equation, get $q_t = \beta$.

Equilibrium

$$\frac{C_{i,t} + q_t C_{i,t+1} = Y_{i,t} + q_t Y_{i,t+1}}{C_{i,t} = \frac{1}{1+\beta}(Y_{i,t} + \beta Y_{i,t+1})} \text{ ?}$$

$$C_{2,t} = C_{2,t+1} = \frac{\beta}{1+\beta}$$

$$C_{1,t} = C_{1,t+1} = \frac{1}{1+\beta}$$

of which the first equation combines budget constraints for two periods.

Combining with $B_{i,t} = \frac{1}{q_t}(Y_{i,t} - C_{i,t})$

$$B_{1,t} = \frac{1}{1+\beta} \quad \text{Saving}$$

$$B_{2,t} = -\frac{1}{1+\beta} \quad \text{borrow}$$

General equilibrium case: Two types of agents and stochastic heterogeneous endowments

- Setup: two future states, p probability of state 1, and $1 - p$ probability of state 2.

$$Y_{1,t+1,1} + Y_{2,t+1,1} = 1$$

2状态

$$Y_{1,t+1,2} + Y_{2,t+1,2} = 1$$

of which the first 1 indicating agent and the last 1 indicating state.

- Budget constraint at period 1:

状态依赖债券(e.g. 保险)

$$C_{i,t} + q_{t,1}B_{i,t,1} + q_{t,2}B_{i,t,2} \leq Y_{i,t}$$

状态1才有效　　状态2才有效

- Budget constraint at period 2:

$$C_{i,t+1,1} \leq Y_{i,t+1,1} + B_{i,t,1}$$

$$C_{i,t+1,2} \leq Y_{i,t+1,2} + B_{i,t,2}$$

Maximization Problem

- Two states: a high state $Y_{t+1,1}$ with probability p, a low state $Y_{t+1,2}$ with probability $1-p$.

$$\max_{C_{i,t},B_{i,t,1},B_{i,t,2},C_{i,t+1,1},C_{i,t+1,2}} lnC_{i,t} + \beta p lnC_{i,t+1,1} + \beta(1-p) lnC_{i,t+1,2}$$

$$s.t. \quad C_{i,t} + q_{t,1}B_{i,t,1} + q_{t,2}B_{i,t,2} \leq Y_{i,t}$$

$$C_{i,t+1,1} \leq Y_{i,t+1,1} + B_{i,t,1}$$

$$C_{i,t+1,2} \leq Y_{i,t+1,2} + B_{i,t,2}$$

Form a Lagrangian:

$$\mathcal{L} = lnC_{i,t} + \beta p lnC_{i,t+1,1} + \beta(1-p) lnC_{i,t+1,2} + \lambda_{i,1}(Y_{i,t} - C_{i,t} - q_{t,1}B_{i,t,1} - q_{t,2}B_{i,t,2}) + \lambda_{i,2}(Y_{i,t+1,1} + B_{i,t,1} - C_{i,t+1,1}) + \lambda_{i,3}(Y_{i,t+1,2} + B_{i,t,2} - C_{i,t+1,2})$$

FOCs

$$\{C_{i,t}\}: \quad \frac{1}{C_{i,t}} = \lambda_{i,1}$$

$$\{C_{i,t+1,1}\}: \quad p\beta \frac{1}{C_{i,t+1,1}} = \lambda_{i,2}$$

$$\{C_{i,t+1,2}\}: \quad (1-p)\beta \frac{1}{C_{i,t+1,2}} = \lambda_{i,3}$$

$$\{B_{i,t,1}\}: \quad q_{t,1}\lambda_{i,1} = \lambda_{i,2}$$

$$\{B_{i,t,2}\}: \quad q_{t,2}\lambda_{i,1} = \lambda_{i,3}$$

We get

$$\frac{p}{1-p}\frac{C_{i,t+1,2}}{C_{i,t+1,1}} = \frac{\lambda_{i,2}}{\lambda_{i,3}}$$

$$\frac{\lambda_{i,2}}{\lambda_{i,3}} = \frac{q_{t,1}}{q_{t,2}}$$

$$\frac{\lambda_{1,2}}{\lambda_{1,3}} = \frac{\lambda_{2,2}}{\lambda_{2,3}}$$

FOCs

$$\frac{p}{1-p}\frac{C_{i,t+1,2}}{C_{i,t+1,1}} = \frac{q_{t,1}}{q_{t,2}}$$

For agents

针对 t+1 的 1, 2 进行分配

$$\frac{C_{1,t+1,2}}{C_{1,t+1,1}} = \frac{C_{2,t+1,2}}{C_{2,t+1,1}}$$

$$\frac{C_{1,t+1,2}}{C_{2,t+1,2}} = \frac{C_{1,t+1,1}}{C_{2,t+1,1}}$$

参考 One agent
针对 t, t+1 进行分配

$$\frac{C_{1,t+1}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{2,t}}$$

$$\frac{C_{2,t}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{1,t+1}}$$

What does this mean? Why is this the case?

- Household consumption problem is to maximize:

$$E_0 \sum_{t=0}^{T} \beta^t U(C_t) = \mathbb{E}_0 [u(C_0) + \beta u(C_1) + \cdots + \beta^T u(C_T)]$$

CES
Constant Elasticity of Substitution

aggregate C_t 替代关系

$$C_t = \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

subject to a budget constraint

- Problem can be divided into two parts:
  - Solve for $c_t(i)$ subject to C_t
  - Solve for C_t

Consumption: Macro vs. Micro

- Expenditure on goods in budget constraint:

$$\int_0^1 p(i)c(i)di$$

- Define P_t as the minimum expenditure needed to purchase one unit of the composite consumption good C_t
- Then it turns out that

$$\int_0^1 p(i)c(i)di = P_t C_t$$

- So, we can write budget constraint without reference to $c(i)$s and $p(i)$s
- P_t is the ideal price index

Motivating Questions

- Social Programs / Taxes / Inequality :
  - Do people save “enough” for retirement?
  - How does consumption respond to an unemployment spell?
  - How much do the super-rich save?
- Business Cycles:
  - How does consumption respond to monetary policy?
  - How does consumption respond to stimulus checks / UI extensions?
- Long-Run Growth:
  - What are the determinants of aggregate savings?

History of Thought

KEYNES' CONSUMPTION FUNCTION

Keynes (1936, p. 96):

> *The fundamental psychological law, upon which we are entitled to depend with great confidence both* a priori *and from our knowledge of human nature and from detailed facts of experience, is that men are disposed, as a rule and on average, to increase their consumption as their income increases, but not by as much as the increase in their income.*

$$\begin{cases} C = \alpha Y + a \\ \Delta C = \alpha \Delta Y, \quad 0 < \alpha < 1. \end{cases}$$

Keynes (1936, p. 93-94): 短期利率波动不会使人们支出大幅改变、

> *The usual type of short-period fluctuation in the rate of interest is not likely, however, to have much* direct *influence on spending either way. There are not many people who will alter their way of living because the rate of interest has fallen from 5 to 4 per cent, if their aggregate income is the same as before.*

Keynes' Consumption Function

Tax

$$C_t = \alpha + \gamma(Y_t - T_t)$$

没有 interest rate

- Consumption a function of after-tax income
- Marginal propensity to consume (γ) between zero and one 边际消费倾向
- Interest rates not important
- Future income not important 未来收入不影响*

Consumption: Introspection

- Suppose you receive a surprise one-time $1,000 scholarship.
- How much would you spend within a year?

CONSUMPTION: INTROSPECTION

- Suppose you receive a surprise one-time $1,000 scholarship.
- How much would you spend within a year?

- How about a one-time $10,000 scholarship?

Consumption: Introspection

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- How about a one-time $10,000 scholarship?
- How about a one-time $100,000 scholarship?

Consumption: Introspection

- Suppose you receive a surprise one-time $1,000 scholarship.
- How much would you spend within a year?

- How about a one-time $10,000 scholarship?
- How about a one-time $100,000 scholarship?

- How about a person like me?

Consumption: Introspection

- Suppose interest rates rose by 1 percentage point.
- How much less would you consume over a year as a fraction of your annual consumption?

Consumption: Introspection

- Suppose interest rates rose by 1 percentage point.
- How much less would you consume over a year as a fraction of your annual consumption?

- Suppose you received news that short term interest rates were going to be 1 percentage point higher (than you thought before) 5 years from now for one year.
- How much less would you consume over this coming year as a fraction of your annual consumption?

Keynesian Cross

- Suppose I, G, NX are exogenous
  (i.e., not functions of output directly or indirectly)
- Planned expenditure (aggregate demand):

$$PE_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

Keynesian Cross

- Suppose I, G, NX are exogenous

  (i.e., not functions of output directly or indirectly)

- Planned expenditure (aggregate demand):

$$PE_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

- Suppose the output is completely demand determined
- Output must equal PE_t:

$$Y_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

- A little algebra then yields:

$$Y_t = \frac{1}{1-\gamma}[\alpha - \gamma T_t + I_t + G_t + NX_t]$$

KEYNESIAN CROSS

i.e. $\Delta G_t = 1$, $\Delta Y_t = \frac{1}{1-\gamma}$.

$$\text{Government purchases multiplier} = \frac{1}{1-\gamma}$$

$\vee$

$$\text{Tax cut multiplier} = \frac{\gamma}{1-\gamma}$$ i.e. $\Delta T_t = -1$, $\Delta Y = \frac{\gamma}{1-\gamma}$.

Tax Effect < Government Purchase Effect

- With MPC = 2/3: G multiplier is 3 and T multiplier is 2
- Logic:
  - Government spends: ΔG (which raises income by ΔG)
  - First round change in consumption: $\gamma \Delta G$
  - Second round change in consumption: $\gamma^2 \Delta G$
  - Etc. $\Delta Y_t = \Delta G_t + \gamma \Delta G_t + \cdots + \gamma^T \Delta G_t + \cdots = (1 + \gamma + \cdots + \gamma^T + \cdots) \Delta G_t = \frac{1}{1-\gamma} \Delta G_t$
- Multiplier for change to "autonomous spending" (i.e., α) same as for G

$\Delta Y = -\Delta T_t (\gamma + \gamma^2 + \cdots + \gamma^T + \cdots) = \frac{\gamma}{1-\gamma} \Delta T_t$.

Keynesian Cross

- Keynesian Economics in its simplest form
- VERY strong assumptions!!
  1. Simplistic consumption function
  2. Investment exogenous
  3. No prices change as output changes
     (i.e., economy completely demand determined)
  4. No monetary policy response
     (but wouldn't matter since nothing responds to interest rate)
- IS-LM: $I(r)$ + monetary policy response
- New Keynesian model: "Modern" consumption function
  + Phillips curve + monetary policy

The Problem of Thrift

- Classical Economics:
    - Saving is good
    - Foundation for capital accumulation
- (Old) Keynesian Economics:
    - Increased saving / fall in "autonomous" spending (i.e., α) thought to have contributed to causing the Great Depression
    - Widespread worry during WWII about "secular stagnation"
        - As people get richer, they will save larger share of income (MPC < 1)
        - Eventually too much saving
        - Not enough demand, not enough investment opportunities

Empirics of Consumption Function

- Early work looked at budget studies
  (i.e., cross section at a point in time)
- $\Delta C / \Delta Y \approx 2/3$

- Also analyzed aggregate saving over course of Great Depression
  - Savings rose as economy recovered

Three Landmark Empirical Studies

"dealt a fatal blow to this extraordinarily simple view of the savings process" (Modigliani 86)

- Simon Kuznetz (1946):
  - National Income and Product Accounts back to 1899
  - No rise in aggregate savings over time
- Dorothy Brady and Rose D. Friedman (1947):
  - Re-analyze budget study data
  - Consumption function shifts up over time as average income increases
- Margaret Reid (unpublished):
  - Re-analyzes budget study data
  - Introduces concept of "permanent component of income"

(See Burns (2022) for history of "Hidden Figures.")

Permanent Income Hypothesis

and
Life-Cycle Hypothesis

PIH / LCH

- Originally developed independently by:
  - Modigiani and Brumberg (1954) (Life-Cycle Hypothesis)
  - Friedman (1957) (Permanent Income Hypothesis)
- Basic idea:
  - Utility maximization and perfect markets imply that current consumption is determined by net present value of life-time income
- Dramatically different from Keynesian consumption function

Simplifying assumptions:

- Known finite horizon T ∞
- No uncertainty
- Constant interest rate $\bar{i}$
- No durable goods (houses/cars/etc) 无耐用品
- Exogenous income process
- Costless enforcement of contracts 无交易成本
- No bankruptcy (i.e., full commitment to repay debt) 银行信用良好
- Natural borrowing limit 自然借贷限制

- Preferences:

$$\sum_{t=0}^{T} \beta^t U(C_t)$$

- Savings/Borrowing technology:
  - Household can save at rate r
  - Household can borrow at rate r up to some limit
  - Household assets denoted A_t 资产 A_t.
- Initial assets: A_{-1} 初始下标 -1
- Income stream: Y_t

Household's Problem

- Maximize

$$\sum_{t=0}^{T} \beta^t U(C_t)$$

- Subject to "budget constraint":

$$\boxed{\frac{A_t}{1+r} + C_t = Y_t + A_{t-1}}$$ 预算约束

  - But, mathematically, this is not really a constraint (doesn't constrain the problem)
  - Mathematically, this is just a definition of A_t

Budget Constraint

Real constraint is constraint on A_t sequence

- Simplest: "Natural" borrowing limit: $A_T \geq 0$ 理性：$A_T = 0$.
  (i.e., household cannot die with debt)

- Alternative: No (unsecured) borrowing: $A_t > 0$
  (much tighter / much more realistic)

INTERTEMPORAL BUDGET CONSTRAINT

- With natural borrowing limit, sequence of one-period budget constraints can be consolidated into a single intertemporal budget constraint:

每个式子/(1+r)后累加

跨期预算约束 $$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

总消费净现值

- Present value of consumption cannot be larger than present value of income and assets

- This embeds the $A_T \geq 0$ constraint

Household's Problem

$$\max \sum_{t=0}^{T} \beta^t U(C_t)$$

subject to:

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

- Important to differentiate between:
  - Choice variables: C_t (and A_t, for $t \geq 0$)
  - Exogenous variables: A_{-1} and Y_t (and r and β)

Quick Note on Readings

- Krusell (2015, ch. 4) is preferred reading on dynamic optimization
  - Hard to strike right balance on technical details (this is not a math class)
  - Sims lecture notes, Stokey, Lucas, with Prescott (1989),
    Ljungqvist and Sargent (2018), Acemoglu (2009) are more techy
  - Romer (2019) is less techy
- Various readings present slightly different versions of the problem
  - E.g., Krusell (2015) presents planner problem with production
  - Good for you to see slight variations in notation and setup

Household's Problem: Solution

- One way to solve household's problem is to set up a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(C_t) - \lambda \left(\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} \right)$$

and derive Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U'(C_t) - \frac{\lambda}{(1+r)^t} = 0$$

- Differentiating yield first order conditions:

$$\boxed{\beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}}$$

- The full set of optimality conditions additionally includes a complementary slackness condition:

$$\lambda \left(\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} \right) = 0$$

Complementary Slackness Condition

$$\lambda\left(\sum_{t=0}^{T}\frac{C_t}{(1+r)^t}-A_{-1}-\sum_{t=0}^{T}\frac{Y_t}{(1+r)^t}\right)=0$$

- Notice from first order conditions that:

$$\lambda=\beta^t(1+r)^tU'(C_t)$$

- If $U'(C_t)>0$, then $\lambda>0$
- Implies that:

$$\sum_{t=0}^{T}\frac{C_t}{(1+r)^t}-A_{-1}-\sum_{t=0}^{T}\frac{Y_t}{(1+r)^t}=0$$

- Since $U'(C_t)>0$, intertemporal budget constraint holds with equality.
- Often we just impose this from the beginning

$$\beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}$$

$$\beta^{t+1} U'(C_{t+1}) = \frac{\lambda}{(1+r)^{t+1}}$$

Divide one by the other:

$$\frac{U'(C_t)}{\beta U'(C_{t+1})} = (1+r)$$

Rearrange:

$$\boxed{U'(C_t) = \beta(1+r)U'(C_{t+1})}$$

This equation is usually referred to as the consumption Euler equation

- Lagrangian math does not yield much intuition
- Alternative: Calculus of variations
- We seek to maximize

$$V(C) = \sum_{t=0}^{T} \beta^t U(C_t)$$

subject to

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

- Here C denotes the sequence $\{C_0, C_1, ..., C_{T-1}, C_T\}$

- Suppose we have a candidate optimal path C_t^*
- Let's consider a variation on this path:
  - Save ϵ more at time t
  - Consume proceeds at time $t+1$
- Utility from new path:

$$V(C) = ... + \beta^t U(C_t^* - \epsilon) + \beta^{t+1} U(C_{t+1}^* + \epsilon(1+r)) + ...$$

- If C_t^* is the optimum, then

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = 0$$

- At the optimum, benefit of small variation must be zero

Calculus of Variations

$$\frac{dV}{d\epsilon} = -\beta^t U'(C_t^* - \epsilon) + (1+r)\beta^{t+1} U'(C_{t+1}^* + \epsilon(1+r))$$

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = -\beta^t U'(C_t^*) + (1+r)\beta^{t+1} U'(C_{t+1}^*) = 0$$

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

- The generic first order condition in calculus of variations is called the Euler equation (or Euler-Lagrange equation)
- This is where the consumption Euler equation gets its name

Permanent Income Hypothesis

- Suppose $U(C_t) = \log C_t$
- Then

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

becomes:

$$\frac{C_{t+1}}{C_t} = \beta(1+r)$$

PERMANENT INCOME HYPOTHESIS

- Suppose $U(C_t) = \log C_t$
- Then

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

becomes:

$$\frac{C_{t+1}}{C_t} = \beta(1+r)$$

- How does consumption growth C_{t+1}/C_t depend on income growth Y_{t+1}/Y_t?

- Suppose $U(C_t) = \log C_t$
- Then

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

becomes:

$$\frac{C_{t+1}}{C_t} = \beta(1+r)$$

- How does consumption growth C_{t+1}/C_t depend on income growth Y_{t+1}/Y_t?
- It doesn't!!

Permanent Income Hypothesis

- Suppose for simplicity that $\beta(1 + r) = 1$
- Consumption Euler equation becomes

$$U'(C_t) = U'(C_{t+1})$$

  which implies

$$C_t = C_{t+1}$$

- Consumers optimally smooth their consumption

Permanent Income Hypothesis

- Suppose for simplicity that $\beta(1+r) = 1$
- Consumption Euler equation becomes

$$U'(C_t) = U'(C_{t+1})$$

which implies

$$C_t = C_{t+1}$$

- Consumers optimally smooth their consumption
- Variation in consumption only due to:
  - Variation in interest rates
  - Variation in marginal utility $U_t'(C_t)$ (e.g., children, health)

Permanent Income Hypothesis

- Let's plug $C_t = C_{t+1}$ into intertemporal budget constraint:

$$C_0 = \Phi(r)\left(A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}\right)$$

$$\Phi(r) = \frac{1 - \frac{1}{1+r}}{1 - \left(\frac{1}{1+r}\right)^{T+1}}$$

- Consumption a function of present value of life-time income
- Current income is not special

$$C_0 = \Phi(r)\left(A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}\right)$$

$$\Phi(r) = \frac{1 - \frac{1}{1+r}}{1 - \left(\frac{1}{1+r}\right)^{T+1}}$$

- Marginal propensity to consume (MPC): How much of a windfall extra dollar a household spends over some period of time
- What does Permanent Income Hypothesis imply about MPC?

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

$$C_0 \sum_{t=0}^{T} \frac{1}{(1+r)^t} = \left(A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}\right)$$

$$C_0 = \frac{1}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}}$$

PERMANENT INCOME HYPOTHESIS 持久收入假设

$$C_0 = \Phi(r)\left(A_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}\right)$$

$$\Phi(r) = \frac{1 - \frac{1}{1+r}}{1 - \left(\frac{1}{1+r}\right)^{T+1}}$$

- Marginal propensity to consume (MPC): How much of a windfall extra dollar a household spends over some period of time
- What does Permanent Income Hypothesis imply about MPC?
- MPC out of windfall gain is equal to $\Phi(r)$
- Suppose T = 40 and r = 0.02:

Keynsian: $\Phi(r) = 0.667$

$$\Phi(r) = 0.035$$

end. ////

Infinite Horizon and Uncertainty

Let's consider a version of the household problem with:

- Infinite horizon
- Uncertainty
- Heterogeneous preferences 偏好多样性

We still maintain:

- No durable goods (houses/cars/etc)
- Exogenous income process
- Costless enforcement of contracts
- No bankruptcy (i.e., full commitment to repay debt)
- Natural borrowing limit

WHY INFINITE HORIZON?

1. Altruism: We love our children
   - If we value our children's consumption like our own, intergenerational discounting is the same as intragenerational discounting
   - If we however value giving (not children's consumption) things are different (warm glow bequests)

2. Simplicity:
   - Infinite horizon makes problem more stationary
   - In finite horizon problem, horizon is a state variable (i.e., affects optimal choice)
   - Solution to problem with long horizon similar to one with infinite horizon

Nature of Uncertainty

- Household *i* faces uncertainty about future income Y_{it+j} (include *i* to emphasize that risk is partly idiosyncratic)
- Heterogeneity in income across households potentially yields heterogeneity in consumption: C_{it}

What Assets Are Traded?

风险厌恶

- If households are risk averse, they will want to "buy insurance" against income risk
- Whether they can depends on what assets are traded
- Two polar cases:
  - 相当于insurance Complete markets: Complete set of state contingent assets available
  - 相当于saving Bonds only: Only non-state contingent asset available
- We will start by considering the complete markets case

完全市场

Natural Borrowing Limit

- What is the “natural” borrowing limit in the infinite horizon case?

Natural Borrowing Limit

- What is the "natural" borrowing limit in the infinite horizon case?
- Household can "borrow" (sell assets) up to the point where it can repay for sure in all states of the world 最坏状况也能偿还
- Rules out "Ponzi schemes":
  - Sell asset at time t
  - Sell more assets at time $t + 1$ to pay off interest/principle coming due
  - Keep doing this ad infinitum
- Natural borrowing limit can be quite "tight":
  - If non-zero probability of zero future income, natural borrowing limit is zero

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U_i(C_{it}(s^t))$$

subject to:

当期消费 (附上)每种可能的资产和 当期收入 初始资产

$$C_{it}(s^t) + \sum_{s^{t+1}|s^t} Q_t(s^{t+1}) A_{it}(s^{t+1}) = Y_{it}(s^t) + A_{i,t-1}(s^t),$$

a No Ponzi scheme constraint, and given $A_{i,-1}$

- $s^t = [s_1, s_2, ..., s_t]$ denotes history of states up to date t t时刻 s的历史轨迹
- $Q_t(s^{t+1})$ denotes the time t price of Arrow security that pays off one unit of consumption in state s^{t+1}
- $A_{it}(s^{t+1})$ denotes quantity of Arrow security that pays off in state s^{t+1} that is purchased at time t by household i

Household's Problem: Solution

- One solution method (again) is to set up a Lagrangian:

$$\mathcal{L}_t = E_t \sum_{j=0}^{\infty} \beta^j \left(U_i(C_{i,t+j}(s^{t+j})) - \lambda_{i,t+j}(s^{t+j}) \left(C_{i,t+j}(s^{t+j}) \right. \right.$$
$$\left. \left. + \sum_{s^{t+j+1}|s^{t+j}} Q_{t+j}(s^{t+j+1}) A_{i,t+j}(s^{t+j+1}) - Y_{i,t+j}(s^{t+j}) - A_{i,t+j-1}(s^{t+j}) \right) \right)$$

- The choice variables at time t are $C_{it}(s^t)$ and $A_{it}(s^{t+1})$

First Order Conditions

- Differentiation of time t Lagrangian yields:

$$\frac{\partial \mathcal{L}_t}{\partial C_{it}(s^t)} : U_i'(C_{it}(s^t)) = \lambda_{it}(s^t)$$

$$\frac{\partial \mathcal{L}_t}{\partial A_{it}(s^{t+1})} : \lambda_{it}(s^t) Q_t(s^{t+1}) = E_t[\beta \lambda_{it+1}(s^{t+1}) I(s^{t+1})]$$

where $I(s^{t+1})$ is an indicator for whether state s^{t+1} occurs

- The latter of these can be rewritten:

$$\lambda_{it}(s^t) Q_t(s^{t+1}) = \beta p_t(s^{t+1}) \lambda_{it+1}(s^{t+1})$$

where $p_t(s^{t+1})$ is the time t probability of state s^{t+1} occurring

- See Sims Lecture Notes for more general “cookbook”

Euler Equation

- Combining equations from last slide to eliminate λ_{it} we get:

$$Q_t(s^{t+1})U_i'(C_{it}(s^t)) = \beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))$$

- This is a version of the consumption Euler equation
  - Trades off consumption today and consumption in one particular state tomorrow
  - Cost today: $Q_t(s^{t+1})U_i'(C_{it}(s^t))$
  - Expected benefit tomorrow: $\beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))$

Euler Equation

- Since Euler equation holds for each state s^{t+1}, it also holds on average

$$
\begin{array}{rrcl}
 & \sum_{s^{t+1}|s_t}\left[Q_t(s^{t+1})U_i'(C_{it}(s^t))\right] & = & \sum_{s^{t+1}|s_t}\left[\beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))\right] \\
=> & U_i'(C_{it}(s^t))\sum_{s^{t+1}|s_t} Q_t(s^{t+1}) & = & \beta E_t[U_i'(C_{it+1}(s^{t+1}))] \\
=> & U_i'(C_{it}(s^t))(1+R_{ft}(s^t))^{-1} & = & \beta E_t[U_i'(C_{it+1}(s^{t+1}))] \\
=> & U_i'(C_{it}(s^t)) & = & \beta(1+R_{ft})E_t[U_i'(C_{it+1}(s^{t+1}))]
\end{array}
$$

where $R_{ft}(s^t)$ is the riskless interest rate in state s^t

- Notice that buying one unit of each Arrow security is the same as buying a riskless bond

Transversality Condition 横截条件

$u'(c_T) \cdot A_t \geq 0 \Rightarrow A_t = 0$

- In finite horizon case, there was a complementary slackness condition that said that household should not die with positive wealth
- What is the counterpart in infinite horizon case?

Transversality Condition

- In finite horizon case, there was a complementary slackness condition that said that household should not die with positive wealth
- What is the counterpart in infinite horizon case?
- Transversality condition:

$$\lim_{j\to\infty} \beta^j E_t[\underbrace{U_i'(C_{it+j}(s^{t+j}))}_{\text{消费边际效用}}\underbrace{A_{it+j}(s^{t+j})}_{\text{资产}}] \leq 0$$

- Intuitively:
  - Cannot be optimal to choose a plan that leaves resources with positive net present value today unspent in the infinite future $\lim_{j\to\infty} A_{it+j}(s^{t+j}) = 0.$
  - Cannot be optimal to allow your wealth to explode at a rate faster than discounted marginal utility is falling $C\uparrow, U'\downarrow$ 财富增速不会超过边际效益减速.
- Are Havard/Princeton/Stanford etc. optimizing?

Transversality vs. No Ponzi

横断条件.　　无庞氏骗局

- Transversality and No Ponzi are sometimes confused
- Very different in nature!!!
- No Ponzi:
  - Debt cannot explode 不能留债
  - Constraint imposed by lenders 债主的强制约束
- Transversality:
  - Wealth cannot explode (too fast) 财富不要增长过快
  - Necessary condition for optimality 最优的必要条件

- Complete markets and common beliefs imply perfect risk sharing
- The consumption Euler equation

$$Q_t(s^{t+1})U_i'(C_{it}(s^t)) = \beta p_t(s^{t+1})U_i'(C_{it+1}(s^{t+1}))$$

holds for all households

- This implies

$$\frac{Q_t(s^{t+1})}{\beta p_t(s^{t+1})} = \frac{U_i'(C_{it+1}(s^{t+1}))}{U_i'(C_{it}(s^t))} = \frac{U_k'(C_{kt+1}(s^{t+1}))}{U_k'(C_{kt}(s^t))}$$

- Taking the ratio of this equation for states s^{t+1} and s^{*t+1} yields

$$\frac{U_i'(C_{it+1}(s^{t+1}))}{U_i'(C_{it+1}(s^{*t+1}))} = \frac{U_k'(C_{kt+1}(s^{t+1}))}{U_k'(C_{kt+1}(s^{*t+1}))}$$

Risk Sharing

$$\frac{U_i'(C_{it+1}(s^{t+1}))}{U_i'(C_{it+1}(s^{*t+1}))} = \frac{U_k'(C_{kt+1}(s^{t+1}))}{U_k'(C_{kt+1}(s^{*t+1}))}$$

- Ratio of marginal utility of all households perfectly correlated
- This is called perfect risk sharing
- See Campbell (2018, ch. 4.1.6)
- Empirical literature: Cochrane (1991), Mace (1991), Schulhofer-Wohl (2010) Townsend (1994), Ogaki and Zhang (2001), Ligon, Thomas, Worrall (2002), Fafchamps and Lund (2003), Mazzucco and Saini (2012)

- Perfect risk sharing condition implies all households have the same ordering of marginal utility and consumption across states
- We can number the states s^{t+1} such that

$$C_{it+1}(1) \leq C_{it+1}(2) \leq ... \leq C_{it+1}(S)$$

- Define $\bar{C}_{t+1}(s^{t+1}) = \sum_i C_{it+1}(s^{t+1})$ and we get

$$\bar{C}_{t+1}(1) \leq \bar{C}_{t+1}(2) \leq ... \leq \bar{C}_{t+1}(S)$$

- We also have

$$\frac{Q_t(1)}{\beta p_t(1)} \geq \frac{Q_t(2)}{\beta p_t(2)} \geq ... \geq \frac{Q_t(S)}{\beta p_t(S)}$$

i.e., assets that provide insurance are expensive

- We can now define a function $g(\bar{C}_{t+1}(s^{t+1}))$ such that

$$\frac{g(\bar{C}_t(s^{t+1}))}{g(\bar{C}_t(s^{*t+1}))} = \frac{Q_t(s^{t+1})/\beta p_t(s^{s+1})}{Q_t(s^{*t+1})/\beta p_t(s^{*t+1})}$$

  for all states s^{t+1} and s^{*t+1}

- $g(\bar{C}_{t+1}(s^{t+1}))$ can be interpreted as the marginal utility of a "composite household" or "representative household"

- We can then integrate to get a function $v(\bar{C}(s^{t+1})$ such that

$$v'(\bar{C}(s^{t+1})) = g(\bar{C}_{t+1}(s^{t+1}))$$

  which is the utility function of the representative household

Representative Household

- Complete market and common beliefs are one way to justify the common representative household assumption
- Important limitations:
  - Utility function of representative household need not be the same as that of individual households
  - Does not generally imply “demand aggregation”: reallocation of wealth alters representative household’s utility function and aggregate demand
  - See Campbell (2018, ch. 4.1.7), Constantinides (1982), Guvenen (2011)
- Demand aggregation requires “Gorman preferences” (see MWG ch. 4.D and Acemoglu (2009, ch. 5.2))
- Representative household assumption is a pretty strong assumption

- Old Keynesian economics:
  - Backward-looking system

$$c_t = \alpha c_{t-1} + \beta y_t$$

The Great Paradigm Shift

- Old Keynesian economics:
  - Backward-looking system

$$c_t = \alpha c_{t-1} + \beta y_t$$

- The most important idea in macroeconomics in the 20th century:

People are forward looking

  - Milton Friedman, Robert Lucas, etc.

THE GREAT PARADIGM SHIFT

- Old Keynesian economics:
  - Backward-looking system 向后看

$$c_t = \alpha c_{t-1} + \beta y_t$$

- The most important idea in macroeconomics in the 20th century:

People are forward looking

  - Milton Friedman, Robert Lucas, etc.

- Pendulum swung really far: 向前看(预期)

$$c_t = E_t c_{t+1} - \sigma(i_t - E_t \pi_{t+1})$$

THE GREAT PARADIGM SHIFT

- Old Keynesian economics:
  - Backward-looking system

$$c_t = \alpha c_{t-1} + \beta y_t$$

- The most important idea in macroeconomics in the 20th century:

People are forward looking

  - Milton Friedman, Robert Lucas, etc.

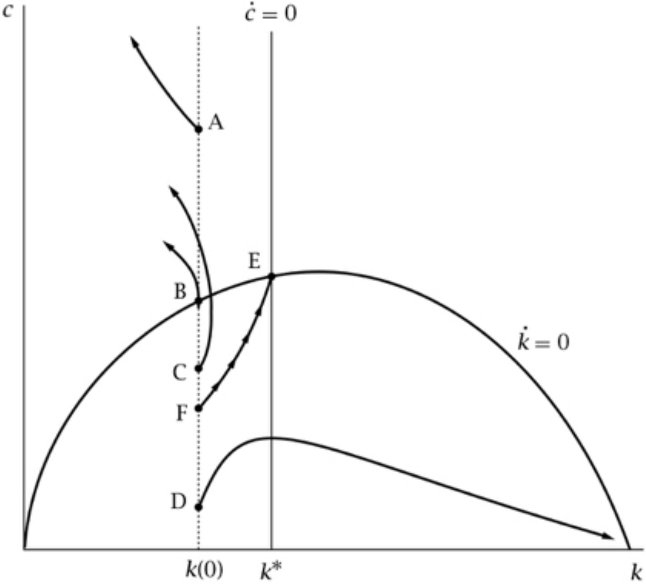
- Pendulum swung really far:

$$c_t = E_t c_{t+1} - \sigma(i_t - E_t \pi_{t+1})$$

- Maybe the world is somewhere in between

Economics versus Physics/Engineering

- Biggest difference: People are forward looking!
- How does this show up mathematically?
  - System does not have a full set of initial conditions
  - Rather, some of the boundary conditions are terminal conditions
- Example:
  - Household problem does not come with an initial condition for consumption
  - The boundary condition is the transversality condition
- Water rolling down a hill is not forward looking: Problem comes with a full set of initial conditions (not a transversality condition)
- Presence of expectations not the fundamental difference. Fundamental difference is nature of boundary conditions.



Source: Romer (2019)

Solving Models

One approach:

1. Solve for first order conditions of each agent's problem (household/firm/etc)
    - System of non-linear dynamic equations (difference or differential equations)
    - E.g., consumption Euler equation, capital accumulation equation, labor supply curve, Phillips curve
    - N equations for N unknown variables for each period t plus a set of boundary conditions
2. Solve this system of non-linear dynamic equations
    - If problem doesn't have "kinks": can linearize (perturbation methods), and solve linear dynamic system (e.g., Blanchard-Kahn algorithm)
    - If problem does have "kinks": Need to use "global methods"

DYNAMIC PROGRAMMING

- Dynamic Programming is an alternative way to solve dynamic optimization problems
- Has its pros and cons versus Lagrangian methods
- For certain types of problems it is the easiest way to go
  - Problems where continuation value is directly used (e.g., Nash bargaining)
  - Problems where non-linearities are important (sometimes)
- For other problems it is more cumbersome than Lagrangian methods
  - E.g., problems that lend themselves to linearization (e.g., Real Business Cycle models, New Keynesian models)
  - Why? Value function is an extra object. Distinction between state variables and control variables extra headache

Dynamic Programming

The Value Function

$$V(X_t) = \max_{C_t, A_t} E_t \sum_{t=j}^{\infty} \beta^t U_i(C_{i,t+j}(s^{t+j}))$$

subject to

$$C_{it}(s^t) + \sum_{s^{t+1}|s^t} Q_t(s^{t+1}) A_{it}(s^{t+1}) = Y_{it}(s^t) + A_{i,t-1}(s^t),$$

a No Ponzi scheme constraint, and given $A_{i,t-1}$

- X_t is a vector of "state variables"
- Figuring out what variables are in X_t is a crucial element of using Dynamic Programming
- The state contains all known variables that affect the household's value function (i.e., the maximized value of the household's objective function).

The State

What is the state in our household problem?

The State

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- Cash on hand at the beginning of the period: $Y_{it}(s^t) + A_{i,t-1}(s^t)$

The State

What is the state in our household problem?

- Cash on hand at the beginning of the period: $Y_{it}(s^t) + A_{i,t-1}(s^t)$
- Any variable that helps forecast future income
  - If $Y_{it}(s^t)$ is i.i.d.: Nothing
  - If $Y_{it}(s^t)$ is AR(1): $Y_{it}(s^t)$
  - if $Y_{it}(s^t)$ is AR(2): $Y_{it}(s^t)$ and $Y_{it-1}(s^{t-1})$
  - Etc.

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    - if $Y_{it}(s^t)$ is AR(2): $Y_{it}(s^t)$ and $Y_{it-1}(s^{t-1})$
    - Etc.
- Current asset prices $Q_t(s^{t+1})$
- Any variables that help forecast future asset prices

  (Sometimes the whole distribution of every agent's wealth)

The State

What is the state in our household problem?

- Cash on hand at the beginning of the period: $Y_{it}(s^t) + A_{i,t-1}(s^t)$
- Any variable that helps forecast future income
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  - If $Y_{it}(s^t)$ is AR(1): $Y_{it}(s^t)$
  - if $Y_{it}(s^t)$ is AR(2): $Y_{it}(s^t)$ and $Y_{it-1}(s^{t-1})$
  - Etc.
- Current asset prices $Q_t(s^{t+1})$
- Any variables that help forecast future asset prices

  (Sometimes the whole distribution of every agent's wealth)

State vector can potentially be quite large and complicated!!

Yoeman Farmer Model

Consider the following yoeman farmer / planner problem:

$$\max_{\{C_s, K_{s+1}\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

subject to

$$C_t + K_{t+1} \leq f(K_t),$$

$C_t \geq 0$, $K_{t+1} \geq 0$, and K_0 given.

- No markets. Yoeman farmer doesn't interact with the outside world.
- $f(K_t) = F(K_t, N) + (1 - \delta)K_t$
- Consumption-savings problem where savings technology is productive capital (seeds)

Yoeman Farmer Model

- If $U'(C_t) > 0$ for all C_t, no resources will be waisted and resource constraint will hold with equality
- In this case we can plug resource constraint into objective function and get

$$\max_{\{0 \leq K_{t+1} \leq f(K_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(K_t) - K_{t+1})$$

  with K_0 given
- What is the state at time 0?

Yoeman Farmer Model

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- In this case we can plug resource constraint into objective function and get

$$\max_{\{0 \leq K_{t+1} \leq f(K_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(K_t) - K_{t+1})$$

  with K_0 given
- What is the state at time 0?
  - Only K_0
  - No exogenous income source, no prices

Value Function for Yoeman Farmer Model

$$V(K_0) = \max_{\{0 \le K_{t+1} \le f(K_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(K_t) - K_{t+1})$$

- Let's now make use of the fact that

$$\max_{x,y} f(x,y) = \max_{y} \{ \max_{x} f(x,y) \}$$

- I.e., we can maximize in steps:
  - First max over x given y (yields a function of y)
  - Then max resulting function of y over y
- In our context we divide the problem into K_1 and K_t for $t > 1$

Bellman Equation

$$V(K_0) = \max_{\{0 \le K_1 \le f(K_0)\}} \left\{ U(f(K_0) - K_1) + \max_{\{0 \le K_{t+1} \le f(K_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t U(f(K_t) - K_{t+1}) \right\}$$

which simplifies to

$$V(K_0) = \max_{\{0 \le K_1 \le f(K_0)\}} \{U(f(K_0) - K_1) + \beta V(K_1)\}$$

and holds for all times t. So, we can write:

$$V(K) = \max_{\{0 \le K' \le f(K)\}} \{U(f(K) - K') + \beta V(K')\}$$

- This equation is called the Bellman equation
- It is a functional equation: the unknown is a function $V(K)$

Policy Function

- The object of primary interest is actually not $V(K)$
- It is the decision rule (policy function):

$$K' = g(K)$$

where

$$g(K) = \arg \max_{\{0 \leq K' \leq f(K)\}} \{U(f(K) - K') + \beta V(K')\}$$

Useful Facts

$$V(K) = \max_{K' \in \Gamma(K)} \{F(K, K') + \beta V(K')\}$$

Suppose:

- F is continuously differentiable in its two arguments, strictly increasing in K, and decreasing in K', strictly concave, and bounded.
- Γ is a nonempty, compact-valued, monotone, and continuous correspondence with a convext graph
- $\beta \in (0, 1)$

Useful Facts

Then:

- There exists a unique $V(K)$ that solves the Bellman equation
- Value function iteration: $V(K)$ can be found as follows:
  - Pick any initial $V_0(K)$
  - Find $V_1(K)$ by evaluating RHS of Bellman equation

  Repeat until $V_{n+1}(K)$ converges to a stable function. This is $V(K)$.
- V is strictly concave
- V is strictly increasing
- V is continuously differentiable
- Optimal behavior can be characterized by a function g: $K' = g(K)$ that is increasing as long as F_2 is increasing in K

For proofs, see Stokey, Lucas, with Prescott (1989)

Extra Readings

Those interested in more detail should read:

- Stokey, Lucas, with Prescott (1989) [starting with ch. 2.1, 3, and 4]
- Ljungqvist and Sargent (2018) [starting with ch. 3 and 4]
  - Note: I do not agree with Ljungqvist and Sargent's strong emphasis on the importance of recursive methods. I think they overdo this.

Finite Horizon Yoeman Farmer Model

$$\max_{\{C_s, K_{s+1}\}_{s=0}^{T}} \sum_{t=0}^{T} \beta^t U(C_t)$$

subject to

$$C_t + K_{t+1} \leq f(K_t),$$

$C_t \geq 0$, $K_{t+1} \geq 0$, and K_0 given.

- Non-stationary problem
- Value function will be different in each period $V_t(K_t)$
- Can be solved by backward induction

Backward Induction

- Start by solving the problem at time T as a function of K_T
- Clearly $K_{T+1} = 0$ is optimal and $V_{T+1}(K_{T+1}) = 0$
- This implies that

$$\begin{aligned} V_T(K_T) &= \max_{\{0 \leq K_{T+1} \leq f(K_T)\}} \{U(f(K_T) - K_{T+1}) + \beta V_{T+1}(K_{T+1})\} \\ &= U(f(K_T)) \end{aligned}$$

Backward Induction

- Then move back one period to $T-1$
- The Bellman function for this period is:

$$V_{T-1}(K_{T-1}) = \max_{\{0 \leq K_T \leq f(K_{T-1})\}} \{U(f(K_{T-1}) - K_T) + \beta V_T(K_T)\}$$

- Since $V_T(K_T)$ is known from prior step, this can be easily solved for $V_{T-1}(K_{T-1})$

Backward Induction

- This process can be iterated backward all the way to t = 0
- Notice that this algorithm is essentially the same as the value function iteration algorithm we discussed for finding $V(K)$ in the infinite horizon case
- This similarity means that the behavior of a household with a long but finite horizon is similar to the behavior of a household with an infinite horizon

Policy Function Iteration

- Start with an initial guess for the policy function $K' = g_0(K)$
- Calculate the value function for this policy function

$$V_0(K) = \sum_{t=0}^{\infty} \beta^t U(f(K) - g_0(K))$$

(In practice a finite sum with a large T)

- Generate a new policy function

$$g_1(K) = \arg\max_{K'} \{U(f(K) - K') + \beta V_0(K')\}$$

- Iterate on this algorithm until the policy function converges

Solving Bellman Equations in Practice

- Four methods:
  1. Guess a solution
  2. Iterate on Bellman equation analytically
  3. Iterate on Bellman equation numerically
  4. Iterate on policy function numerically
- First two methods only work for highly special models
- In practice, Dynamic Programming most useful for problems that require numerical solution methods

Functional Euler Equation

- We can derive an Euler equation from the Bellman function

$$V(K) = \max_{K' \in \Gamma(K)} \{F(K, K') + \beta V(K')\}$$

- Using the policy function $K' = g(K)$ we can rewrite the Bellman equation:

$$V(K) = F(K, g(K)) + \beta V(g(K))$$

- Also, this policy function satisfies the first order condition

$$F_2(K, K') + \beta V'(K') = 0$$

- Evaluating this equation at $K' = g(K)$ we get

$$F_2(K, g(K)) + \beta V'(g(K)) = 0$$

Functional Euler Equation

- If we differentiate

$$V(K) = F(K, g(K)) + \beta V(g(K))$$

with respect to K we get

$$V'(K) = F_1(K, g(K)) + g'(K)\{F_2(K, g(K)) + \beta V'(g(K))\}$$

- Second term on RHS is zero (see last slide) and we get:

$$V'(K) = F_1(K, g(K))$$

- The fact that the second term drops out is an application of the envelope theorem

Functional Euler Equation

- Combining:

$$F_2(K, g(K)) + \beta V'(g(K)) = 0$$

$$V'(g(K)) = F_1(g(K), g(g(K)))$$

yields

$$F_2(K, g(k)) = -\beta F_1(g(K), g(g(K)))$$

- This is the Euler equation stated as a functional equation

Functional Euler Equation

- In yoeman farmer model:

$$
\begin{aligned}
F(K, K') &= U(F(K) - K') = U(C(K)) \\
F_1(K, K') &= U'(C(K))F'(K) \\
F_2(K, K') &= -U'(C(K))
\end{aligned}
$$

- This means that the Euler equation in the yoeman farmer model is

$$U'(C(K)) = \beta F'(K')U'(C(K'))$$

- $F'(K)$ plays the role of the return on investment
- $C(K) = F(K) - g(K)$ is the yoeman farmer's consumption function

Functional Euler Equation

$$U'(C(K)) = \beta F'(K')U'(C(K'))$$

- Iterating on the functional Euler equation for $C(K)$ is an alternative to value function iteration, policy function iteration, and Blanchard-Kahn methods
- Advantages:
  - Level of value function not important (only derivative). Value function iteration waists one polynomial point on getting the level
  - Computationally useful in models with many agents and many frictions (see McKay's notes)
- Disadvantage:
  - May not converge

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